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LEG and RS filtering problems — A bit more general setting
open questions

Filtering with Exponential Criteria
Discrete time case

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LEG problem in one slide

The model

- nonobservable signal sequence (X_t), $t \geq 1$ with values in $\mathbb{R}^1$;
- observations (Y_t) from $\mathbb{R}^1$;
- exponential type payoff function L_T

The aim

To find $\bar{h}$ which minimizes the payoff function.

LEG problem in one slide

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LEG Filtering Problem

The precise statement

observation model

signal– (X_t), observations— (Y_t): $(X_t,\ Y_t)_{t\geq 1}$ is Gaussian.

Aim: To minimize with respect to h : $h_t \in \mathcal{Y}_t, t \geq 1$ the quantity:

Exponential Criterium,I

$$\frac{1}{\mu} \ln \left[\mathbb{E} \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\} \right],$$

▸ A Quiz

where

- h : h_t is $\mathcal{Y}_t$-measurable, $\mathcal{Y}_t = \sigma(\{Y_u\,,\ 1 \leq u \leq t\})$
- Q_s, $1 \leq s \leq T$: given nonnegative numbers

Filtering with Exponential Criteria

Le Breton, Viot, Kleptsyna

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Three different cases

There are three different cases for LEG filtering problem:

- $\mu = 0$ - risk-neutral filtering problem.
- $\mu > 0$ - risk-averse filtering problem.
- $\mu < 0$ - risk-preferring filtering problem.

Our approach

- Solve the problem for $\mu < 0$ (it is easier).
- Reduce the problem to an auxiliary risk-neutral filtering problem.
- Extend results to the general case using the analytical properties.

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The second problem: Risk Sensitive Filtering

Recursive equation as a definition of the Risk-sensitive Filtering:

Exponential Criterium,II

$$\hat{g}_t = \arg\min_{g \in \mathcal{Y}_t} \frac{1}{\mu} \ln \Big[\mathbb{E} \exp \Big\{ \frac{\mu}{2} (X_t - g)^2 Q_t + \frac{\mu}{2} \sum_{s=1}^{t-1} (X_s - \hat{g}_s)^2 Q_s \Big\} \Big/ \mathcal{Y}_t \Big],$$

where g is a $\mathcal{Y}_t$ measurable variable.

RS problem, result

Connection between two problems

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Are they equal?

Q: Can we always take $\bar{h} = \hat{h}$?

A: Sometimes **yes**, sometimes **no** ...

Connection between two problems

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A Quiz,I

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short memory

Q: What happens for a short memory criterium?

i.i.d.signal

Q: What happens if X i.i.d.?

A Quiz,I

short memory

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A Quiz,II

Quadratic Criterium

$$L_T(h,\mu) = \mathbb{E}\left[\frac{\mu}{2}\sum_0^T (X_s - h(s))^2 Q_s\right],$$

Risk - Neutral Filtering

Q: What happens for the quadratic type payoff function?
A: Solutions of LQG and RS: $g = \bar{h}_T$, $\bar{h}_t = \hat{h}_t = \pi_t(X)$, where $\pi_t(X) := \mathbb{E}[X_t|\mathcal{Y}_t]$ (can be computed using Kalman filter).

A Quiz,II

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Kalman filter,I

AR(1) Markov model

- **signal** $X_t = a_t X_{t-1} + D_t^{\frac{1}{2}} \widetilde{\varepsilon}_t\,,\ t \geq 1\,; \quad X_0 = x\,,$
- **observation**:

$$Y_t = A_t X_t + \varepsilon_t\,, \quad t \geq 1\,.$$

where

- $\varepsilon = (\varepsilon_t)_{t \geq 1}$ - i.i.d. $N(0,1)$ random variables, independent of X;
- $A := (A_t,\ t \geq 1)$ some sequence of the real numbers.

Kalman filter,I

Estimation

$$\pi_t(X) = a_t\pi_{t-1}(X) + \frac{A_t\overline{\gamma}_t}{1 + A_t^2\overline{\gamma}_t}[Y_t - a_tA_t\pi_{t-1}(X)],\ t \geq 1,\ \pi_0 = x.$$

Filtering error

$$\overline{\gamma}_s = D_s + \frac{a_s^2\overline{\gamma}_{s-1}}{1 + A_{s-1}^2\overline{\gamma}_{s-1}},\ s \geq 1,\ \mathbb{E}[(X_t - \pi_t(X))^2/\mathcal{Y}_t] = \bar{\gamma}(t).$$

Kalman filter,I

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Kalman filter,II

Generalized Kalman filter

$$\pi(X)_t = m_t + \sum_{l=1}^{t} A_l \bar{\gamma}(t,l)(Y_l - A_l \pi_l(X)),$$

Filtering error

$$\bar{\gamma}(t,s) = \Gamma(t,s) - \sum_{l=1}^{s-1} \bar{\gamma}(t,l)\bar{\gamma}(s,l) \frac{A_l^2}{1 + A_l^2 \bar{\gamma}_l}$$

the variance of the filtering error—- $\mathbb{E}[(X_t - \pi_t(X))^2/\mathcal{Y}_t] = \bar{\gamma}(t,t)$.

Kalman filter,II

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Motivations

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- Robust estimation
- H_∞ estimations
- Estimation of probability to exceed the fixed level
- Theory of a system failure. Estimation of the parameters of a survival function with unobservable component.

References

Markov observation model.

- History 1: control & partial observations P.Whittle,1981; A. Bensoussan & J.H. van Schuppen, 1985
- History 2: LEG filtering, discrete time setting J.L. Speyer, 1992 Discrete time Markov observation model
- History 3: Risk-Sensitive setting R.J. Elliott, S. Dey, J.B.Moore, 1994 Risk-Sensitive Filtering, definition by recursive equation
- History 4 "Information State" approach, first definitions proposed by R.J. Elliott, S. Dey, J.B.Moore and ... for RS filtering problem

AR(1) model, J.L. Speyer, 1992

Observation model

$$\begin{cases} X_t = a_t X_{t-1} + D_t^{\frac{1}{2}} \widetilde{\varepsilon}_t \,, \; t \geq 1 \,; \quad X_0 = x \,, \\ Y_t = A_t X_t + \varepsilon_t . \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} \overline{h}_t = a_t \overline{h}_{t-1} + \dfrac{A_t \overline{\gamma}_t}{1 + A_t^2 \overline{\gamma}_t} [Y_t - a_t A_t \overline{h}_{t-1}], \; t \geq 1, \; \overline{h}_0 = x \\ \overline{\gamma}_s = D_s + \dfrac{a_s^2 \overline{\gamma}_{s-1}}{1 + (A_{s-1}^2 - \mu Q_{s-1}) \overline{\gamma}_{s-1}}, \; s \geq 1, \; \overline{\gamma}_0 = 0. \end{cases}$$

Of course, $\overline{h}_t \neq \pi_t(X)$, but may be?

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Remaining question

What to do if the observation model is not Markovian?

references

1. M.L. Kleptsyna, A. Le Breton and M.Viot
SIAM J. Optimization and Control **47** (6) (2008), 2886 - 2911.
2. M.L. Kleptsyna, A. Le Breton and M.Viot
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LEG Filtering Problem, Result

Consider $\bar{h} = (\bar{h}(t),\ t \geq 0)$: solution of LEG filtering problem

LEG, definition

$$\bar{h} = \operatorname{argmin}_{h:\, h_t \in \mathcal{Y}_t, t \geq 1} \frac{1}{\mu} \ln \left[\mathbb{E} \exp \left\{ \frac{\mu}{2} \sum_{t=1}^{T} (X_t - h_t)^2 Q_t \right\} \right].$$

LEG, characterization

$$\bar{h}_t = \hat{\pi}_t(X_t)$$

—the conditional expectation of X w.r.to the new measure

-Which measure?

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-Which measure?

LEG problem, result for Markov type observations

Observations

$$Y_t = A_t X_t + \varepsilon_t\,, \quad t \geq 1\,.$$

LEG, solution

$$\overline{h}_t = m_t + \sum_{l=1}^{t} A_l \overline{\gamma}(t,l)(Y_l - A_l \overline{h}_l),$$

finding γ: Riccati type equation

$$\overline{\gamma}(t,s) = \Gamma(t,s) - \sum_{l=1}^{s-1} \overline{\gamma}(t,l)\overline{\gamma}(s,l)\,\frac{S_l}{1+S_l\overline{\gamma}_l}, \quad S_l = A_l^2 - \mu Q_l$$

RS problem, result

Consider $\hat{h} = (\hat{h}(t),\ t \geq 0)$: solution of RS filtering problem

▸ The second problem: Risk Sensitive Filtering

RS, solution, characterization

$$\hat{h}_t = \widehat{\pi}_t(X_t)$$

- the conditional expectation of X with respect to the new measure.

we have also the equality of two solutions $\hat{h} = \bar{h}$.

RS problem, result

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New measure, the first approach

Change of measure

notation

$$J_t = \exp\left\{ -\frac{1}{2}\sum_{s=1}^{t}(X_s - h_s)^2 Q_s \right\}.$$

New measure

$$\frac{d\,\widehat{\mathbb{P}}}{d\,\mathbb{P}} = \prod_{t=1}^{T} \frac{M_t}{M_{t-1}},$$

with

$$\frac{M_t}{M_{t-1}} = \frac{\pi_{t-1}[\mathbf{1}(X_t \in dx,\ Y_t \in dy) J_{t-1}]}{\pi_{t-1}[J_{t-1}]\mathbb{E}_{t-1}\mathbf{1}(X_t \in dx,\ Y_t \in dy)}\Bigg|_{x=X_t,\, y=Y_t}.$$

New measure-property

Property

- with respect to the new measure $\widehat{\mathbb{P}}$ variables $(X_t), t \geq 1$ are independent
- Y_t does not depend on $(X_s), s \leq t-1$.
- $$\pi_t[J_t] = \widehat{\pi}_t[J_t]\pi_t[M_t]$$

it is not exactly the classical Bayes formula.

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Back to the initial measure - calculation rules
Auxiliary filtering problem

Auxiliary observations, I

$\bar{Y}_t = (Y_t^1, Y_t^2)$, such that

$$\begin{cases} Y_t^1 = Y_t, \\ Y_t^2 = Q_t(X_t - h_t) + \sqrt{Q_t}\bar{\varepsilon}_t, \end{cases}$$

where $\bar{\varepsilon} = (\bar{\varepsilon}_t)_{t\geq 1}$ – a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables independent of X

Auxiliary observations, II

$\xi_t = \sum_{s=1}^{t}(X_s - h_s)Y_s^2$.

Bayes formula

Back to the initial measure - calculation rules

Auxiliary filtering problem

Auxiliary observations, I

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$\xi_t = \sum_{s=1}^{t}(X_s - h_s)Y_s^2.$

Bayes formula

Back to the initial measure

Link between measures

- $\widehat{\pi}_t(X) = \bar{\pi}_{t,t-1}(X_t) - \bar{\gamma}_{X\xi}(t, t-1)$
  with
  - $\bar{\pi}_{t,t-1}(X) = \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}]$ –the conditional expectation of X
  - $\bar{\gamma}_{X\xi}(t, t-1) =$ $\mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}])(\xi_{t-1} - \bar{\pi}_{t-1}(\xi_{t-1}))/\bar{\mathcal{Y}}_{t,t-1}]$ - the conditional covariance
- $\widehat{\gamma}_t = \mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}])]^2$ — the variance of the filtering error
- σ-field $\mathcal{Y}_{t,t-1} = \sigma(\{(Y_s, Y_r^2), 1 \le s \le t, 1 \le r \le t-1\})$.

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  - $\bar{\pi}_{t,t-1}(X) = \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}]$ –the conditional expectation of X
  - $\bar{\gamma}_{X\xi}(t, t-1) = \mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}])(\xi_{t-1} - \bar{\pi}_{t-1}(\xi_{t-1}))/\bar{\mathcal{Y}}_{t,t-1}]$ - the conditional covariance
- $\widehat{\gamma}_t = \mathbb{E}[(X_t - \mathbb{E}[X_t/\bar{\mathcal{Y}}_{t,t-1}])]^2$ — the variance of the filtering error
- σ-field $\mathcal{Y}_{t,t-1} = \sigma(\{(Y_s, Y_r^2), 1 \leq s \leq t, 1 \leq r \leq t-1\})$.

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Applying our solution to particular cases

Methodology

- Write equations for
  - the variance of the filtering error $\bar{\gamma}_{xx}(t, t-1)$;
  - for the difference $\widehat{\pi}_t(X) = \bar{\pi}_{t,t-1}(X) - \bar{\gamma}_{X\xi}(t)$.
- Solve the LEG filtering problem: take $\bar{h} = \widehat{\pi}_t(X)$.

Notation

$(\varepsilon_t, \widetilde{\varepsilon}_t,\ t = 1, 2, \dots)$ is a sequence of i.i.d. standard Gaussian random variables

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AR(1) model

Observation model

$$\begin{cases} X_t = a_t X_{t-1} + D_t^{\frac{1}{2}} \tilde{\varepsilon}_t\,, \; t \geq 1\,; \quad X_0 = x\,, \\ Y_t = A_t X_t + \varepsilon_t. \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} \overline{h}_t = a_t \overline{h}_{t-1} + \frac{A_t \overline{\gamma}_t}{1+A_t^2 \overline{\gamma}_t} [Y_t - a_t A_t \overline{h}_{t-1}],\; t \geq 1,\; \overline{h}_0 = x., \\ \overline{\gamma}_s = D_s + \frac{a_s^2 \overline{\gamma}_{s-1}}{1+(A_{s-1}^2 - \mu Q_{s-1})\overline{\gamma}_{s-1}},\; s \geq 1,\; \overline{\gamma}_0 = 0. \end{cases}$$

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MA(1) model

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$$\begin{cases} X_t = \widetilde{\varepsilon}_t + \lambda \widetilde{\varepsilon}_{t-1}\,;\, t \geq 1\,, \\ Y_t = A_t X_t + \varepsilon_t. \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} \overline{\gamma}_t = 1 + \lambda^2 - \lambda \frac{A_{t-1}^2 - \mu Q_{t-1}}{1+(A_{t-1}^2 - \mu Q_{t-1})\overline{\gamma}_{t-1}}\,,\ t \geq 1\,; \quad \overline{\gamma}_0 = 1 + \lambda^2, \\ \overline{h}_t = \lambda \frac{A_{t-1}}{1+A_t^2 \overline{\gamma}_t}[Y_{t-1} - A_{t-1}\overline{h}_{t-1}] + \frac{A_t \overline{\gamma}_t}{1+A_t^2\overline{\gamma}_t} Y_t,\ t \geq 1,\ \overline{h}_0 = x. \end{cases}$$

MA(1) model

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"Non Markov" observations, can be elaborated

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Two problems, again

For given positive symmetric deterministic 2×2 matrices $\Omega_s, 1 \leq s \leq T$, let us set $\Phi_t(h) = (X_t\, h_t)\Omega_t \begin{pmatrix} X_t \\ h_t \end{pmatrix}$.

"LEG setting"

$$\bar{h} = \arg \min_{h_t \in \mathcal{Y}_t,\, t \geq 1} \frac{1}{\mu} \ln \left[\mathbb{E} \left\{ \exp \left\{ \frac{\mu}{2} \sum_1^T \Phi_s(h) \right\} \right\} \right].$$

"RS setting"

$$\hat{h}_t = \arg \min_{g \in \mathcal{Y}_t} \frac{1}{\mu} \ln \left(\mathbb{E} \left[\exp \left\{ \frac{\mu}{2} \Phi_t(g) + \frac{\mu}{2} \sum_1^{t-1} \Phi_s(\hat{h}) \right\} \Big/ \mathcal{Y}_t \right] \right),$$

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Equality of two solutions, yes

The question

Does the equality $\bar{h} = \hat{h}$ hold ?

One possible answer

Yes for degenerated matrices Ω :
$\Omega_{1,1} = \Omega_{2,2} = -\Omega_{1,2} = -\Omega'_{2,1} = Q.$

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LEG problem, solution

$$\Omega = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ A = 1,\ \mu = -1 \text{ and } X_t = X_{t-1} + \tilde{\varepsilon}_t.$$

$$\bar{h}_1 = \frac{1 + \Gamma(T,1)}{2 + \Gamma(T,1)} Y_1$$

where

$$\Gamma(T,t) = 10 \frac{\lambda^T - \lambda^t}{(1-\sqrt{5})\lambda^T - (1+\sqrt{5})\lambda^t},\ \lambda = \frac{(3-\sqrt{5})}{(3+\sqrt{5})}.$$

RS problem, solution

$$\hat{h}_1 = \frac{\pi_1(X_1)}{1+\gamma_1} = \frac{1}{4} Y_1$$

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Open questions

A couple of unexplored cases

1. The non-linear setting
2. Equality of the solutions of the two problems for non-Gaussian models