Asymptotic expansion for multiple integrals

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joint work with Ivan Nourdin and Giovanni Peccati

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Classical problem

- Let $(X_k)_{k\in\mathbb{Z}}$ be a d-dimensional centered stationary Gaussian process with covariance function

$$r^{ij}(l) = \mathbb{E}[X_1^{(i)} X_{1+l}^{(j)}], \qquad 1 \leq i, j \leq d.$$

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- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function and write

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k).$$

Under which conditions on f and r do we have that

$$S_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2) \ ?$$

Central limit theorem [Arcones94]

- Assume that $\mathbb{E}[f(X_1)] = 0$ and let $q \geq 1$ be the Hermite rank of f, i.e.
  (a) $E[f(X_1)p_m(X_1)] = 0$ for all polynomials p_m of degree $< q$,
  (b) $E[f(X_1)p_q(X_1)] \neq 0$ for some polynomial p_q of degree q.

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  (b) $E[f(X_1)p_q(X_1)] \neq 0$ for some polynomial p_q of degree q.
- **Theorem:** Let $E[f^2(X_1)] < \infty$ and assume that

$$\sum_{l \in \mathbb{Z}} |r^{ij}(l)|^q < \infty, \qquad 1 \leq i,j \leq d.$$

Then $\sigma^2 = E[f^2(X_1)] + 2\sum_{l=1}^{\infty} E[f(X_1)f(X_{1+l})] < \infty$ and

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k) \xrightarrow{\mathcal{L}} S \sim \mathcal{N}(0, \sigma^2).$$

Quantitative Breuer-Major theorems

- Problem: Provide a Berry-Esseen relation of the type

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \leq \varphi_n \to 0$$

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Quantitative Breuer-Major theorems

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- We give an explicit bound φ_n for three different classes of functions:

$$h \in C_b^2(\mathbb{R}), \qquad h \in \text{Lip}(1), \qquad h(x) = 1_{(-\infty,z]}(x).$$

Our method is based on a combination of Malliavin calculus, Stein's method and an interpolation technique.

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- Then we show that

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \le \varphi_n$$

with $\varphi_n \to 0$ being given as

$$\varphi_n = \text{Function}\Big(c_h,\ E[f^2(X_1)],\ \sum_{l\in\mathbb{Z}} \theta^q(l),\ \sum_{|l|>n} \theta^q(l),\ \sum_{|l|\le n} \theta^q(l)\frac{|l|}{n}\Big).$$

Main result: the Hermite case

- **Theorem:** Let $d = 1$ and $f = H_q$, where H_q is the qth Hermite polynomial, i.e. $H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}(e^{-x^2/2})$. When $\theta = \sum_{l \in \mathbb{Z}} |r(l)|^q < \infty$ we deduce

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \le c_h(A_n + B_n),$$

where c_h is a constant depending on whether $h \in C_b^2(\mathbb{R})$, $h \in \mathrm{Lip}(1)$ or $h(x) = 1_{\{-\infty,z\}}(x)$. The quantities A_n, B_n are given by

$$A_n = \frac{q!}{2}\theta \left(\sum_{|l| \le n} |r(l)|^q \frac{|l|}{n} + \sum_{|l|>n} |r(l)|^q \right),$$

$$B_n = \frac{1}{2q} \sum_{k=1}^{q-1} k k! \binom{q}{k}^2 \sqrt{(2q-2k)!} \sqrt{\frac{2\theta}{n} \sum_{|l| \le n} |r(l)|^k \sum_{|l| \le n} |r(l)|^{q-k}}$$

Example: fBm

- Let $d = 1$, $f = H_q$ and let $X_i = B_i^H - B_{i-1}^H$ be the fractional Brownian noise with Hurst parameter $H \in (0,1)$. In this case $|r(l)| \sim l^{2H-2}$ as $l \to \infty$, and the CLT
$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_q(X_k) \xrightarrow{\mathcal{L}} S \sim \mathcal{N}\Big(0, q! \sum_{l \in \mathbb{Z}} r(l)^q\Big)$$
holds provided $H \in (0, 1 - \frac{1}{2q})$.

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- Furthermore, when $H \in (0, 1 - \frac{1}{2q})$ we deduce that

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \leq \text{const} \times \begin{cases} n^{-1/2} : & a \in (-2, -1] \\ n^{a/2} : & a \in [-1, -\frac{1}{q-1}] \\ n^{\frac{aq+1}{2}} : & a \in [-\frac{1}{q-1}, -\frac{1}{q}) \end{cases}$$

with $a = 2H - 2$. This result extends to $d(S_n, S)$ with $d = d_{Kol}, d_W$.

Isonormal Gaussian Processes

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathbb{H}$ be a separable Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. A centered Gaussian family $W = \{W(h) | \ h \in \mathbb{H}\}$ is called an isonormal Gaussian process iff

$$E[W(h)W(g)] = \langle h, g \rangle_{\mathbb{H}} \ , \qquad \forall \ h, g \in \mathbb{H}.$$

  Here we assume that $\mathcal{F}$ is generated by W. Note that the map $h \to W(h)$ is linear.

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- Usually the Hilbert space $\mathbb{H}$ is induced by a Gaussian process. Let $(X_k)_{k \in \mathbb{Z}}$ be our original (1-dimensional) stationary Gaussian process. Then $\mathbb{H}$ is induced by

$$r(k-l) = \mathbb{E}[X_k X_l] =:< \mathbf{k}, \mathbf{l} >_{\mathbb{H}} \ .$$

Malliavin derivative

- Set $\mathcal{S} = \{g(W(h_1), \ldots, W(h_n)) |\ g \in C_b^\infty(\mathbb{R}),\ n \geq 1,\ h_i \in \mathbb{H}\}$. For $F = g(W(h_1), \ldots, W(h_n)) \in \mathcal{S}$, define

$$DF = \sum_{i=1}^{n} \frac{dg}{dx_i}(W(h_1), \ldots, W(h_n))h_i \in \mathbb{L}^2(\Omega; \mathbb{H}).$$

$D^{1,2}$ = completion of $\mathcal{S}$ w.r.t.

$$\|F\|_{1,2}^2 = \mathbb{E}F^2 + \mathbb{E}[||DF||_{\mathbb{H}}^2].$$

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- *Example*: Let $F = H_q(W(h))$, $h \in \mathbb{H}$. Then

$$DF = qH_{q-1}(W(h))h.$$

Skorokhod integral

- The Skorokhod integral δ is defined as an (unbounded) adjoint operator of D. Domδ = all elements $u \in \mathbb{L}^2(\Omega; \mathbb{H})$ with

$$|\mathbb{E}[\langle DF, u\rangle_{\mathbb{H}}]| \leq c\|F\|_{\mathbb{L}^2(\Omega)} \qquad \forall F \in D^{1,2}.$$

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$$|\mathbb{E}[\langle DF, u\rangle_{\mathbb{H}}]| \leq c\|F\|_{\mathbb{L}^2(\Omega)} \qquad \forall F \in D^{1,2}.$$
- For $u \in \text{Dom}\delta$, $\delta(u) \in \mathbb{L}^2(\Omega)$ is given by
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- *Example*: Let $u = Fh$ with $F \in \mathcal{S}$ and $h \in \mathbb{H}$. Then

$$\delta(u) = FW(h) - \langle DF, h\rangle_{\mathbb{H}}.$$

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$$\delta(u) = FW(h) - \langle DF, h\rangle_{\mathbb{H}}.$$

- *Useful property*:

$$\delta D[H_q(W(h))] = qH_q(W(h)).$$

Case: $h \in \mathrm{Lip}(1)$

- Let us consider the case $d = 1$, $f = H_q$ and $h \in \mathrm{Lip}(1)$. Hence,

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} H_q(X_k) \xrightarrow{\mathcal{L}} S \sim \mathcal{N}(0, \sigma^2),$$

and we want to bound $|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]|$.

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- Assume wlog that $\sigma^2 = 1$. Let s_h be solution of the Stein's equation associated with h, i.e. s_h solves

$$h(x) - \mathbb{E}[h(S)] = s_h'(x) - x s_h(x).$$

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- By the Stein's equation we rewrite the problem as

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| = |\mathbb{E}[s_h'(S_n) - S_n s_h(S_n)]|.$$

Case: $h \in \mathrm{Lip}(1)$

- As $\|s_h'\|_\infty \le \|h'\|_\infty \le 1$ we deduce that

$$
\begin{aligned}
|\mathbb{E}[s_h'(S_n) - S_n s_h(S_n)]| &= |\mathbb{E}[s_h'(S_n) - q^{-1}\delta D(S_n) s_h(S_n)]| \\
&= |\mathbb{E}[s_h'(S_n) - q^{-1}\langle D(S_n), D(s_h(S_n))\rangle_{\mathbb{H}}]| \\
&= |\mathbb{E}[s_h'(S_n)(1 - q^{-1}\|DS_n\|_{\mathbb{H}}^2)| \\
&\le \|h'\|_\infty \cdot \|(1 - q^{-1}\|DS_n\|_{\mathbb{H}}^2)\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}
$$

A direct computation shows that $\|(1 - q^{-1}\|D(S_n)\|_{\mathbb{H}}^2)\|_{\mathbb{L}^2(\Omega)} \le A_n + B_n \to 0$.

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- The above calculation provides an alternative proof of [Nualart,Ortiz-Latorre08], who proved that

$$S_n \xrightarrow{\mathcal{L}} S \quad \Longleftrightarrow \quad \|(1 - q^{-1}\|DS_n\|_{\mathbb{H}}^2)\|_{\mathbb{L}^2(\Omega)} \to 0.$$

Case: $h \in C_b^2(\mathbb{R})$

- We consider again the case $d = 1$, $f = H_q$ and $\sigma^2 = 1$. Let

$$\Psi(t) = \mathbb{E}[h(\sqrt{1-t}S_n + \sqrt{t}S)], \qquad t \in [0,1].$$

be the interpolation between S_n and S. Then

$$\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)] = \Psi(1) - \Psi(0) = \int_0^1 \Psi'(t)dt.$$

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- We have that

$$\begin{aligned}\Psi'(t) &= \frac{1}{2\sqrt{t}}\mathbb{E}[h'(\sqrt{1-t}S_n + \sqrt{t}S)S] \\ &- \frac{1}{2\sqrt{1-t}}\mathbb{E}[h'(\sqrt{1-t}S_n + \sqrt{t}S)S_n].\end{aligned}$$

Case: $h \in C_b^2(\mathbb{R})$

- Assume wlog that S_n and S are independent. Integration by parts gives

$$\frac{1}{2\sqrt{t}}\mathbb{E}[h'(\sqrt{1-t}S_n + \sqrt{t}S)S] = \frac{1}{2}\mathbb{E}[h''(\sqrt{1-t}S_n + \sqrt{t}S)]$$

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- For the other term we use again the identity $\delta D(S_n) = qS_n$:

$$\frac{1}{2\sqrt{1-t}}\mathbb{E}[h'(\sqrt{1-t}S_n + \sqrt{t}S)S_n]$$

$$= \frac{1}{2q\sqrt{1-t}}\mathbb{E}[h'(\sqrt{1-t}S_n + \sqrt{t}S)\delta D(S_n)]$$

$$= \frac{1}{2q}\mathbb{E}[h''(\sqrt{1-t}S_n + \sqrt{t}S)\|DS_n\|_{\mathbb{H}}^2].$$

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- Now we put things together:

$$\begin{aligned} |\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| &= |\Psi(1) - \Psi(0)| \\ &\leq \sup_{t\in[0,1]} |\Psi'(t)| \\ &\leq \frac{\|h''\|_\infty}{2} \cdot \|(1 - q^{-1}\|DS_n\|_{\mathbb{H}}^2)\|_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

This completes the proof. □

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This completes the proof. □

- A similar result can be shown for $h(x) = 1_{(-\infty,z]}(x)$ using again Stein's method.

Thank you!