# The Symbol of an Itô Process and its Relations to Fine Properties

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# Notation

- Vectors in $\mathbb{R}^d$ are column vectors.
- $x'$ denotes a transposed vector.
- $B_b(\mathbb{R}^d)$ bounded Borel measurable functions
- $C_\infty(\mathbb{R}^d)$ continuous and vanishing at infinity
- $C_c^\infty(\mathbb{R}^d)$ test functions

A stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ is always 'in the background'.

# Outline

1. On Itô Processes
2. Introducing: The Symbol
3. Applications of the Symbol
   - General Applications
   - Indices
   - Example: Lévy Driven SDE

# Itô Processes

## Definition

Itô processes are strong Markov processes, which are as well semimartingales with characteristics of the form

$$\begin{aligned} B_t(\omega) &= \int_0^t \ell(X_s(\omega))\, ds && , \ell(\cdot) \in \mathbb{R}^d \\ C_t(\omega) &= \int_0^t Q(X_s(\omega))\, ds && , Q(\cdot) \text{ pos. semidefinite} \\ \nu(\omega; dt, dy) &= N(X_t(\omega), dy) \times dt && , N(\cdot, dy) \text{ Lévy measure} \end{aligned}$$

with respect to a fixed truncation function $\chi$ for every $\mathbb{P}^x$ $(x \in \mathbb{R}^d)$. We restrict ourselves to $\ell, Q, \int (1 \wedge y^2) N(\cdot, dy)$ which are (finely) continuous and locally bounded.

# Overview

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Lévy | $\subset$ | (rich) Feller | $\subset$ | Itô | $\subset$ | Hunt semimartingale | $\subset$ | Markov semimartingale |
| | | $\cap$ | | | | $\cap$ | | $\cap$ |
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# The Symbol of a Process

## Definition

Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued normal, Markov semimartingale, which is conservative. For $x, \xi \in \mathbb{R}^d$ we define

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t - x)'\xi} - 1}{t}$$

and call $p : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ the **symbol** of the process, if the limit exists.

# The Symbol of a Process

## Definition

Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued normal, Markov semimartingale, which is conservative. For $x, \xi \in \mathbb{R}^d$ we define

$$p^\sigma(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t^\sigma - x)'\xi} - 1}{t}$$

and call $p^\sigma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ the **symbol** of the process, if the limit exists.

where $\sigma$ is the first exit time of an arbitrary compact set containing $x$.

# The Symbol of a Lévy Process

Let $X = (X_t)_{t \geq 0}$ be a Lévy process:

$$
\begin{aligned}
-\lim_{t\downarrow 0} \mathbb{E}^x \frac{e^{i(X_t - x)'\xi} - 1}{t} &= -\lim_{t\downarrow 0} \frac{\mathbb{E}^x(e^{i(X_t-x)'\xi}) - 1}{t} \\
&= -\lim_{t\downarrow 0} \frac{\mathbb{E}^0(e^{iX_t'\xi}) - 1}{t} \\
&= -\lim_{t\downarrow 0} \frac{e^{-t\psi(\xi)} - 1}{t} \\
&= -\frac{\partial^+}{\partial t}\bigg|_{t=0} \left(e^{-t\psi(\xi)}\right) \\
&= \psi(\xi)
\end{aligned}
$$

# The Symbol of a Feller Process

In the case of Feller processes we have

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} p(x,\xi) \widehat{u}(\xi)\, d\xi$$

if it is bounded in the sense

$$q(x,\xi) \le c \cdot (1 + \|\xi\|^2) \qquad \forall x \in \mathbb{R}^d. \tag{G}$$

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In the case of Itô processes the symbol $p^\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is given by

$$p^\sigma(x,\xi) = -\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left( e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y) \right) N(x, dy)$$

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$$\mathbb{E}^0\left(e^{i\xi' X_t}\right) = e^{-t\cdot\psi(\xi)}$$

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$$-\lim_{t\downarrow 0}\mathbb{E}^x\frac{e^{i(X_t^\sigma - x)'\xi}-1}{t}$$

# Applications of the Symbol

Having calculated the symbol it is possible to obtain both: the generator of the process and the semimartingale characteristics. In the case of rich Feller processes satisfying the growth condition $(G)$ in $x$ it is known that properties of the process can be expressed via analytic properties of the symbol, e.g.

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- conservativeness
- $\gamma$-variation
- growth and Hölder conditions
- smoothness of the paths.

# Indices of Symbols

Out of the representation (for $\rho \in \mathbb{R}^d \setminus \{0\}$)

$$\frac{\|y\|^2}{1+\|y\|^2} = \int_{\rho \neq 0} \Big(1 - \cos(y'\rho)\Big) g(\rho)\ d\rho$$

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and define (for $R > 0$)

$$H(x,R) := \sup_{\|y-x\|\leq 2R} \sup_{\|\varepsilon\|\leq 1} \left( \int_{-\infty}^{\infty} \operatorname{Re} p\left(y, \frac{\rho\varepsilon}{R}\right) g(\rho)\ d\rho + \left| p\left(y, \frac{\varepsilon}{R}\right) \right| \right)$$

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and furthermore

$$\beta_\infty^x := \inf \left\{ \lambda > 0 : \limsup_{R \to 0} R^\lambda H(x,R) = 0 \right\}.$$

# A Characterization of the Index $\beta_\infty^x$

## Theorem

For every $x \in \mathbb{R}^d$ such that $\xi \longmapsto p(x,\xi)$ is not identically zero, the index $\beta_\infty^x$ can be calculated in the following way

$$\beta_\infty^x = \limsup_{\|\eta\|\to\infty} \sup_{\|y-x\|\leq 2/\|\eta\|} \frac{\log|p(y,\eta)|}{\log\|\eta\|}.$$

# The Symbol of the Solution of an SDE

## Starting point

Let $Z$ be a $n$-dim. Lévy process and consider the following SDE:

$$\begin{aligned} dX_t &= \Phi(X_{t-})\, dZ_t \\ X_0 &= x \end{aligned} \qquad (\star)$$

where

- $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ is Lipschitz continuous,
- $Z = (Z_t)_{t\geq 0}$ takes its values in $\mathbb{R}^n$,
- The solution $X = (X_t)_{t\geq 0}$ is $\mathbb{R}^d$-valued.

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**Remark:** This SDE has a unique solution. If $\Phi$ is bounded, the solution is a Feller process.

# Calculating the Symbol

Consider (one dimensional):

$$e^{i(X_t^\sigma - x)\xi} - 1 \overset{[\text{Itô}]}{=}$$

$$\int_{0+}^{t} \left( i\xi \cdot e^{i(X_{s-}^\sigma - x)\xi} \right) \, dX_s^\sigma + \frac{1}{2} \int_{0+}^{t} \left( -\xi^2 e^{i(X_{s-}^\sigma - x)\xi} \right) \, d[X_s^\sigma, X_s^\sigma]^c$$

$$+ e^{-ix\xi} \sum_{0<s\leq t} \left( e^{iX_s^\sigma \xi} - e^{iX_{s-}^\sigma \xi} - i\xi e^{iX_{s-}^\sigma \xi} \Delta X_s^\sigma \right)$$

# Calculating the Symbol

Consider (one dimensional):

$$\frac{1}{t}\mathbb{E}^x\left(e^{i(X_t^\sigma - x)\xi} - 1\right) \overset{[\text{Itô}]}{=}$$

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# The Result

We obtain in the case of the solution of an SDE driven by a Lévy process:

$$
\begin{aligned}
p^\sigma(x,\xi) &= -i\ell'(\Phi(x)'\xi) + \frac{1}{2}(\Phi(x)'\xi)'Q(\Phi(x)'\xi) \\
&\quad - \int_{y\neq 0} \left( e^{i(\Phi(x)'\xi)'y} - 1 - i(\Phi(x)'\xi)'y \cdot \chi(y) \right) N(dy)
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where $\psi$ is the symbol of the driving term and $\Phi$ is the coefficient of the SDE.

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$$\begin{aligned} p^\sigma(x,\xi) &= -i\ell'(\Phi(x)'\xi) + \frac{1}{2}(\Phi(x)'\xi)'Q(\Phi(x)'\xi) \\ &\quad - \int_{y\neq 0} \left( e^{i(\Phi(x)'\xi)'y} - 1 - i(\Phi(x)'\xi)'y \cdot \chi(y) \right) N(dy) \\ &= \psi(\Phi(x)'\xi) \end{aligned}$$

where $\psi$ is the symbol of the driving term and $\Phi$ is the coefficient of the SDE.

# An Existence Result

## Corollary

For a given function $p(x,\xi)$ which can be written as

$$p(x,\xi) = \psi(\Phi(x)'\xi)$$

where $\psi$ is continuous negative definite and $\Phi$ is Lipschitz continuous, there exists a corresponding Itô process, which is Feller if $\Phi$ is bounded.

# Technical Main Result

## Theorem

Let $X$ be a solution process of the SDE

$$\begin{aligned} dX_t &= \Phi(X_{t-})\, dZ_t \\ X_0 &= x \end{aligned} \qquad (\star)$$

with $d = n$ and where the linear mapping $\xi \mapsto \Phi(y)'\xi$ is bijective for every $y \in \mathbb{R}^d$. If the driving Lévy process has the non-constant symbol $\psi$ and index $\beta_\infty^\psi$, then the solution $X$ of the SDE has, for every $x \in \mathbb{R}^d$, the index $\beta_\infty^x \equiv \beta_\infty^\psi$.

# A Result on the Strong $\gamma$-Variation of the Process

For $\gamma \in ]0,\infty[$ and a $\mathbb{R}^d$-valued function $g$ on the interval $[a,b]$ we call

$$V^\gamma(g;[a,b]) := \sup_{\pi_n} \sum_{j=1}^{n} \|g(t_j) - g(t_{j-1})\|^\gamma$$

the **(strong)** $\gamma$-**variation** of $g$ on $[a,b]$. The supremum is taken over all partitions $\pi_n = (a = t_0 < t_1 < ... < t_n = b)$.

# A Result on the Strong $\gamma$-Variation of the Process

## Corollary 1

Let $X = (X_t)_{t\geq 0}$ be the solution of the SDE $(\star)$ Under the assumptions of the theorem above for every $\gamma > \beta_\infty^\psi$ the $\gamma$-variation of the process $X$ is on every compact time interval $[0, T]$ a.s. finite.

# A Result on Hölder Conditions

## Corollary 2

Let $X = (X_t)_{t \geq 0}$ be the solution of the SDE $(\star)$. Then

$$\lim_{t \to 0} t^{-1/\lambda} (X_{\cdot} - x)^*_t = 0 \text{ a.s. if } \lambda > \sup_x \beta^x_\infty$$

Under the assumptions of the theorem above, $\sup_x \beta^x_\infty$ is the index of the driving Lévy process: $\beta^\psi_\infty$.

# A Result on Besov Spaces

## Corollary 3

Let $X = (X_t)_{t \geq 0}$ be the solution of the SDE $(\star)$. Under the assumptions of the theorem above we have almost surely

$$\{t \mapsto X_t\} \in B_q^{s,\mathrm{loc}}(L^p(dt)) \quad \text{if} \quad s \cdot \sup_y \{p, q, \beta_\infty^y\} < 1$$

and

$$\{t \mapsto X_t\} \notin B_q^{s,\mathrm{loc}}(L^p(dt)) \quad \text{if} \quad sp > 1.$$

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- Statistics for the symbol.

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Thank you for your attention!

# Feller Processes

## Definition

A **Feller process** is an $\mathbb{R}^d$-valued Markov process $X = (X_t)_{t\geq 0}$ with associated semigroup $(T_t)_{t\geq 0}$ having the following properties:

(F1) $T_t : C_\infty(\mathbb{R}^d) \longrightarrow C_\infty(\mathbb{R}^d)$ for every $t \geq 0$

(F2) $\lim_{t\downarrow 0} \| T_t u - u\|_\infty = 0$ for every $u \in C_\infty(\mathbb{R}^d)$

## The Generator of a Lévy Process

A Lévy process is strongly Markovian and homogeneous in time (and space). Therefore the

$$T_t u(x) = \mathbb{E}^x(u(X_t)), \qquad u \in C_\infty(\mathbb{R}^d),\ t \geq 0,\ x \in \mathbb{R}^d$$

form a semigroup $(T_t)_{t\geq 0}$ of operators on $C_\infty(\mathbb{R}^d)$.

If $(X_t)_{t\geq 0}$ is a Lévy process

then the generator $A$ can be written in the following form (for $u \in C_c^\infty(\mathbb{R}^d)$)

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi}\psi(\xi) \quad \widehat{u}(\xi)\ d\xi$$

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If $(X_t)_{t\geq 0}$ is a Feller process and the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $D(A)$ of the generator then the generator $A$ can be written in the following form (for $u \in C_c^\infty(\mathbb{R}^d)$)

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$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} q(x,\xi) \widehat{u}(\xi)\, d\xi$$

where $q(x,\xi)$ is for every $x$ a continuous negative definite function in the co-variable $\xi$.

# The Generator of a Feller Process

If $(X_t)_{t\geq 0}$ is a Feller process then the generator $A$ has a representation as follows

$$Au(x) = -\int_{\mathbb{R}^d} e^{ix'\xi} q(x,\xi) \widehat{u}(\xi)\, d\xi,$$

where the **symbol** $q : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ can be written as

$$q(x,\xi) = -i\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0} \left( e^{iy'\xi} - 1 - iy'\xi \cdot \chi(y) \right) N(x,dy)$$

with (for every $x \in \mathbb{R}^d$)

- $\ell(x) \in \mathbb{R}^d$,
- $Q(x)$ is a positive semidefinite $d \times d$-matrix,
- $N(x,\cdot)$ is a measure on $\mathbb{R}^d \backslash \{0\}$ such that $\int (1 \wedge y^2)\, N(x,dy) < \infty$,
- $\chi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a cut-off function.